# Delay Robustness and AQM in Networks

X. Fan, S. Kalyanaramanb, M. Arcak and Y. S. Chan

*Abstract*— **This paper studies the stability problem of a network fluid model with transmission and queuing delays in the forward and backward channels. We present a novel small gain approach to prove global asymptotic stability for a network with arbitrary time delays and network routing matrix. Specifically, we use a logarithmic state transformation suggested recently in the literature, and establish a linear input-to-state gain for this transformed system. With the new state variables, the gain of the routing matrix is unity and, thus, the stability condition is scalable and independent of routing. Unlike the previously reported results that employ the logarithmic transformation, we give a simple small-gain interpretation for the delay robustness of the networks. We demonstrate that this small-gain technique is generally applicable for the design of a new class of edge-based active queue management (AQM) algorithms where zero steady queuing delay is achieved. We also show that it can be generalized to the study of other networks, such as the uplink power control problem in CDMA systems.**

*Keywords*— **Keyword one, Keyword two**

## I. INTRODUCTION

In this paper, we consider a general topology network consisting of an arbitrary number of sources and links, which are interconnected through the forward routing matrix $R_f$ and backward routing matrix $R_b$ as shown in Figure 1 and studied in Kelly [1], Low and Lapsley [2], and Kunniyur and Srikant [3].

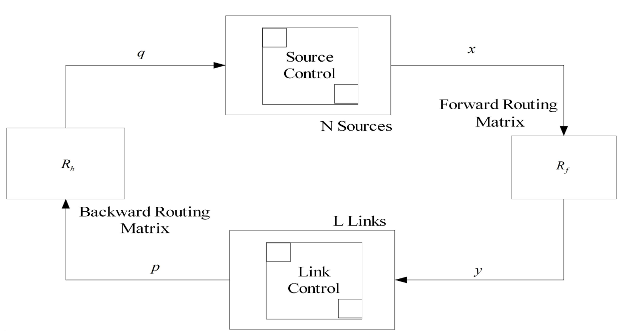


Fig. 1. Network Flow Control Model

Packets from source $i$ (with sending rate $x_i$) are routed through the links with the aggregate link rate $y = R_f x$. We assume that each link $l$ has a fixed capacity $c_l$, and based on its congestion and queue size, a link price $p_l$ is generated:

$$p_l = h\left(y_l, c_l\right), \quad l = 1, \cdots, L. \tag{1}$$

The link price information $p_l$ is then fed back to source with the aggregate source price, $q = R_b^T p$. If there are no transmission and queuing delays, we have $R_f = R_b =: R$ due to the fact that the links only feed back the price information to the sources that utilize them [1]. In the presence of delays, we have,

$$y_l\left(t\right) = \sum_{i \in L(l)} x_i\left(t - \tau_{il}^f\right), \tag{2}$$

$$q_i\left(t\right) = \sum_{l \in T(i)} p_l\left(t - \tau_{il}^b\right), \tag{3}$$

where $L\left(l\right)$ denotes the set of sources which traverse link $l$ and $T\left(i\right)$ is set of links which source $i$ traverses. The quantity $\tau_{il}^f$ represents the forward delay from the source $i$ to link $l$ while the quantity $\tau_{il}^b$ corresponds to the backward delay from link $l$ to source $i$. Hence $R_f$ (respectively, $R_b$) can be obtained from $R$ by multiplying $e^{-\tau_{il}^f s}$ (or $e^{-\tau_{il}^b s}$) with the $il$-th element of matrix $R$. Source $i$ then uses the feedback aggregate price $q_i$ to update its sending rate $x_i$. In this paper, we first consider the following TCP-like congestion control algorithm, which is a generalization of the one proposed in [4]

$$\dot{x}_i\left(t\right) = \left(\kappa_i g_i\left(x_i\left(t\right), x_i\left(t - T_i\right)\right)\left(\frac{1}{x_i^{\alpha}(t)} - q_i(t)\right)\right)_{x_i}^{+}, \quad i = 1, \cdots, N. \tag{4}$$

Let $T_i = \tau_{il}^f + \tau_{il}^b$ denote the round trip delay for source $i$ and $(\cdot)^+$ denote the projection which forces $x_i$ to stay within its physically allowable range $[0, x_{i\max}]$. Kelly showed that (1)-(4) is a specific distributed solution for a static optimization problem, where the goal is to maximize a network utilization function while complying with the capacity constraints in the links $Rx \leq c$.

Our main interest in this formulation is that it encompasses a broad class of transportation control protocols in the Internet, and can be extended to other applications, such as the power control problems in CDMA networks [5]. A common feature in these applications is that, feedback signals are generated at remote locations, and transmitted to the actuators via communication channels [5]. Transmission and queuing delays can thus be significant, and threaten the stability properties predicted by the delay-free models.

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The rest of the paper is organized as follows. In Part One, we present a novel small gain approach to prove that systems (1)-(4) remain globally asymptotically stable for arbitrary time delays and network routing matrix when a specific class of price functions are chosen for (1). The main idea is that we use a logarithmic state transformation suggested recently in [6], and then establish a linear input-to-state gain [7] for the transformed system. With the new state variables, the gain of the routing matrix can be shown to be unity. Hence, the stability condition is scalable and independent of routing matrix. Unlike the previously reported results that employ the logarithmic transformation [6], we give a simple small-gain interpretation for the delay robustness of networks.

In Part Two of the paper, we employ the small-gain technique to design a new class of edge-based AQM algorithms to achieve zero steady-state queue-length and hence zero queuing delay in the network. As in the adaptive virtual queue (AVQ) [8] designs, we regulate the aggregate rate $y_l$ of each link $l$ to render it slightly less than the link capacity $c_l$. The difference of our design however is that, instead of implementing queue management at every router, we only require AQM be implemented at the edge router, thus avoiding costly upgrade of the core network. Specifically, in our design we regard every edge-to-edge flow path as a virtual link with virtual demand which leads to a path congestion signal. We let the edge router detect this congestion signal and send a corresponding price back to source. We prove that, with this extra feedback, this edge-based AQM is globally asymptotically stable with zero steady-state queuing delay. In the analysis, we employ a logarithmic state transformation generalized from [6], and show that the extra feedback satisfies an input-to-state stability (ISS) gain with respect to transformed states of the nominal system [7]. We then proceed to derive conditions under which the same addition satisfies a complementary ISS gain with respect to the price fed back to the source.

We conclude the paper with the extension of this small-gain technique to the delay robustness of a gradient algorithm proposed in [9], [10] for CDMA uplink power control. Compared with the earlier study employing a passivity framework for this problem in [11], the results in this paper are delay-independent and do not rely on the channel gains.

*1) Notation and Definitions:* a) We denote by $\|\cdot\|_p$ the $p$-norm of vectors and induced $p$-norm of matrices. Whenever the choice of $p$ is unimportant, we drop the subscript. We further assume $\|x\|_{L_p}$ is the $L_p$-norm of $x(t) = [x_1(t), \cdots, x_i(t), \cdots, x_N(t)]^T$ on the interval $[0, \infty)$, $p \in (0, \infty)$. When $p = \infty$,

$$\|x(t)\|_{L_\infty} = \sup_{t \geq 0} \|x(t)\|_\infty = \sup_{t \geq 0} \max_{i \leq N} |x_i(t)|. \quad (5)$$

For $x \in L_\infty$, we define $\|x\|_a = \lim_{t\to\infty} \sup \|x(t)\|$.

b) A function $\gamma(\cdot) : \mathrm{R}_{\geq 0} \to \mathrm{R}_{\geq 0}$ is defined to be class-$K$ if it is continuous, zero at zero, and strictly increasing. It is said to be class-$K_\infty$ if it is class-$K$and grows unbounded.

c) A system $\dot{x} = f(x, u)$ is said to be *input-to-state stable* (ISS) if there exist class-$K_\infty$ functions $\gamma_0(\cdot)$ and $\gamma(\cdot)$ such that, for any inputs $u(\cdot) \in L_\infty^m$ and $x_0 \in R^n$, the response $x(t)$ from the initial state $x(0) = x_0$ satisfies

$$\|x\|_{L_\infty} \leq \max\left\{\gamma_0(\|x_0\|),\ \gamma\left(\|u\|_{L_\infty}\right)\right\}, \quad \|x\|_a \leq \gamma(\|u\|_a).$$

The function $\gamma(\cdot)$ is referred to as the ISS gain.

d) Let $C([-r, 0], \mathrm{R}^n)$ be the Banach space of continuous functions mapping the interval $[-r, 0]$ into $\mathrm{R}^n$, with the topology of uniform convergence. For the time-delay system

$$\dot{x} = f(x_t) \quad (6)$$

where $x_t := x(t + \theta),\ -r \leq \theta \leq 0$, with the initial condition

$$x(t) = \psi(t),\ t \in [-r, 0],\ \ \psi \in C([-r, 0], \mathrm{R}^n), \quad (7)$$

the equilibrium $x = 0$ is said to be *globally asymptotically stable* if for any $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ such that $\|\psi\|_{L_\infty} \leq \delta$ implies $\|x\|_{L_\infty} \leq \varepsilon$, and for all $\psi \in C([-r, 0], \mathrm{R}^n)$, $x(t) \to 0$ as $t \to \infty$.

## II. Main Result

In this section, we present our main result (Theorem 1) and, in the next section, we illustrate how the result can be used to design a new class of edge-based AQM algorithms. We specify the price function as

$$p_l = \left(\frac{y_l}{c_l}\right)^\beta,\ \ l = 1, \cdots, L. \quad (8)$$

which implies the probability that an M/M/1 queue with an arrival rate $y_l$ and capacity $c_l$ is of length $\beta$ or greater [4]. Other price functions will be discussed in the next section.

*Theorem 1:* Consider the delay feedback system (1)-(4) with the price function (8) and suppose

$$g_i(u, v) \geq \Delta, \qquad \forall u, v \geq 0 \text{ and } \forall i = 1, \cdots, N \quad (9)$$

for some $\Delta > 0$. If

$$\beta < \alpha \quad (10)$$

then the network is globally asymptotically stable for arbitrary delays $\tau_{il}^f$ and $\tau_{il}^b$.

Proof: Let

$$\tilde{x}_i = \ln\frac{x_i}{x_i^*},\ \tilde{q}_i = \ln\frac{q_i}{q_i^*},\ \tilde{p}_l = \ln\frac{p_l}{p_l^*},\ \tilde{y}_l = \ln\frac{y_l}{y_l^*}, \quad (11)$$

where $x_i^* > 0$ is the equilibrium of source sending rates $x_i$ and so are $q_i^*$, $p_i^*$ and $y_i^*$. Without loss of generality, we shift the equilibrium $x_i = x_i^*$ to $\tilde{x}_i^* = 0$. Thus, the system (1)-(4), with the new states variables, becomes

$$\begin{aligned} \tilde{p}_l &= \ln\tfrac{p_l}{p_l^*} = \beta\ln y_l - \beta\ln c_l - \beta\ln y_l^* + \beta\ln c_l \\ &= \beta\ln\tfrac{y_l}{y_l^*} = \beta\tilde{y}_l \end{aligned} \quad (12)$$

$$\dot{\tilde{x}}_i(t) = \kappa_i\frac{g_i(x_i(t), x_i(t - T_i))}{x_i^{\alpha+1}(t)}\left(1 - e^{\tilde{q} + \alpha\tilde{x}_i(t)}\right) \quad (13)$$

Due to space limitations, we have dropped the projection $(\cdot)^+$ on the right hand side of (4) and omit the discussion on the discontinuity issues due to the projection. However, the result still holds when this discontinuity is taken into consideration. We prove the result in three steps: In *Step 1*, we show that there exists a linear ISS gain from $\tilde{q} = \begin{bmatrix} \tilde{q}_1 & \tilde{q}_2 & \cdots & \tilde{q}_N \end{bmatrix}^T$ to $\tilde{x} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 & \cdots & \tilde{x}_N \end{bmatrix}^T$. In *Step 2*, we prove the existence of a complementary ISS-gain from $\tilde{x}$ to $\tilde{q}$. In *Step 3*, we combine the results of Steps 1 and 2, and prove the stability of the system by employing the ISS Small-Gain Theorem [12], [13].

*Step 1:* For the subsystem (13), we choose $V_i(\tilde{x}_i) = \frac{1}{2}\tilde{x}_i^2$ as the ISS Lyapunov function [7], and get its derivative as

$$\dot{V}_i = \tilde{x}_i\dot{\tilde{x}}_i = \kappa_i \frac{g_i(x_i(t), x_i(t-T_i))}{x_i^{\alpha+1}(t)}\tilde{x}_i\left(1 - e^{\tilde{q}_i(t)+\alpha\tilde{x}_i(t)}\right). \tag{14}$$

Since for any $0 < \lambda < 1$, when $|\tilde{q}_i| \le (1-\lambda)\alpha|\tilde{x}_i|$,

$$\dot{V}_i \le \kappa_i \frac{\Delta}{x_{i\_\max}^{\alpha+1}} \frac{|\tilde{x}_i|}{e^{\lambda|\tilde{x}_i|}}\left(1 - e^{\lambda\alpha|\tilde{x}_i|}\right). \tag{15}$$

we claim from [7] and [12] that

$$|\tilde{x}_i(t)| \le \gamma(|\tilde{x}_i(0)|, t) + \frac{1}{(1-\lambda)\alpha}|\tilde{q}_i|_{L_\infty}, \tag{16}$$

where $\gamma(\cdot,\cdot)$ is a class-$KL$ function. Therefore, the $L_\infty-$ and asymptotic norm of $\tilde{x}_i$ are bounded in the ISS sense:

$$\|\tilde{x}_i(t)\|_{L_\infty} \le \max\left\{\|\tilde{x}_i(0)\|, \frac{1}{(1-\lambda)\alpha}\|\tilde{q}_i(t)\|_{L_\infty}\right\}, \tag{17}$$

$$\|\tilde{x}_i(t)\|_a \le \frac{1}{(1-\lambda)\alpha}\|\tilde{q}_i\|_a. \tag{18}$$

Recalling the definition of $L_\infty$ norm of the vector $x(t)$ in (5), we also obtain the ISS bound of the vector $\tilde{x}$ as

$$\|\tilde{x}(t)\|_{L_\infty} \le \max\left\{\|\tilde{x}(0)\|, \gamma_1\|\tilde{q}\|_{L_\infty}\right\}, \tag{19}$$

$$\|\tilde{x}(t)\|_a \le \gamma_1\|\tilde{q}\|_a, \tag{20}$$

where

$$\gamma_1 = \frac{1}{(1-\lambda)\alpha}. \tag{21}$$

*Step 2:* We prove the complementary ISS-gain from $\tilde{x}$ to $\tilde{q}$ is $\beta$ by showing the ISS-gains from $\tilde{x}$ to $\tilde{y}$, from $\tilde{y}$ to $\tilde{p}$, and from $\tilde{p}$ to $\tilde{q}$ one by one, since the price feedback $\tilde{q}$ can be regarded as the output of the above three cascaded subsystems. Among these three gains, the gain from $\tilde{y}$ to $\tilde{p}$ is straightforward from (12) and equal to $\beta$. To find the ISS gain from $\tilde{x}$ to $\tilde{y}_l$, we substitute (2) and $y_l^* = \sum\limits_{i\in L(l)} x_i^*$ into $\tilde{y}_l = \ln\frac{y_l}{y_l^*}$, and get

$$\tilde{y}_l = \ln\frac{y_l}{y_l^*} = \ln\frac{\sum\limits_{i\in L(l)} x_i\left(t-\tau_{il}^f\right)}{\sum\limits_{i\in L(l)} x_i^*}. \tag{22}$$

Since

$$\ln\frac{\sum\limits_{i\in L(l)} x_i}{\sum\limits_{i\in L(l)} x_i^*} \le \ln\max\left\{\max_{i\in L(l)}\left\{\frac{x_i}{x_i^*}\right\}, 1\right\}$$
$$= \max\left\{\max_{i\in L(l)}\ln\left\{\frac{x_i}{x_i^*}\right\}, 0\right\} = \max\left\{\max_{i\in L(l)}\tilde{x}_i, 0\right\} \tag{23}$$

and

$$\ln\frac{\sum\limits_{i\in L(l)} x_i}{\sum\limits_{i\in L(l)} x_i^*} \ge \ln\min\left\{\min_{i\in L(l)}\left\{\frac{x_i}{x_i^*}\right\}, 1\right\}$$
$$\ge \min\left\{\min_{i\in L(l)}\ln\left\{\frac{x_i}{x_i^*}\right\}, 0\right\} = \min\left\{\min_{i\in L(l)}\tilde{x}_i, 0\right\} \tag{24}$$

the absolute value of $\ln\frac{\sum\limits_{i\in L(l)} x_i}{\sum\limits_{i\in L(l)} x_i^*}$ is bounded:

$$\begin{aligned}\left|\ln\frac{\sum\limits_{i\in L(l)} x_i}{\sum\limits_{i\in L(l)} x_i^*}\right| &\le \max\left\{\left|\max_{i\in L(l)}\tilde{x}_i\right|, \left|\min_{i\in L(l)}\tilde{x}_i\right|\right\} \\ &= \max_{i\in L(l)}|\tilde{x}_i| \le \max_{i\le N}|\tilde{x}_i|\end{aligned} \tag{25}$$

Thus the $L_\infty$- and asymptotic norms of $y_l$ are

$$\begin{aligned}\|\tilde{y}_l(t)\|_{L_\infty} &\le \sup_{0\le t<\infty}\left(\max_{i\le N}\left|\tilde{x}_i\left(t-\tau_{il}^f\right)\right|\right) \\ &\le \sup_{0\le t<\infty}\|\tilde{x}(t)\| + \sup_{-\tau\le t<0}\|\tilde{x}(t)\| = \|\tilde{x}\|_{L_\infty} + \|\psi_1\|_{L_\infty}\end{aligned} \tag{26}$$

$$\|\tilde{y}_l\|_a \le \|\tilde{x}(t)\|_a \tag{27}$$

where $\tau = \max\limits_{i\le N, l\le L}\left\{\tau_{il}^f\right\}$ and $\psi_1$ is the initial condition of $\tilde{x}(t)$ as defined in (7). Recalling the definition of the vector's $L_\infty$-norm in (5), we obtain

$$\|\tilde{y}\|_{L_\infty} \le \|\tilde{x}\|_{L_\infty} + \|\psi_1\|_{L_\infty}, \quad \|\tilde{y}\|_a \le \|\tilde{x}\|_a \tag{28}$$

and claim that the ISS-gains from $\tilde{x}$ to $\tilde{y}$ is unit. Similarly, the ISS-gains from $\tilde{p}$ to $\tilde{q}$ is unit and

$$\|\tilde{q}\|_{L_\infty} \le \|\tilde{p}\|_{L_\infty} + \|\psi_2\|_{L_\infty}, \quad \|\tilde{q}\|_a \le \|\tilde{p}\|_a \tag{29}$$

where $\psi_2$ is the initial condition of $\tilde{p}(t)$. Combining the linear gain $\beta$ from $\tilde{y}$ to $\tilde{p}$ in (12), we show that the complementary ISS-gain from $\tilde{x}$ to $\tilde{q}$ is $\beta$, and specifically,

$$\begin{aligned}&\|\tilde{q}\|_{L_\infty} \le \beta\left(\|\tilde{x}\|_{L_\infty} + \|\psi_1\|_{L_\infty} + \|\psi_2\|_{L_\infty}\right) \\ &\|\tilde{q}\|_a \le \beta\|\tilde{x}\|_a\end{aligned} \tag{30}$$

*Step 3:* Combining the ISS gain $\gamma_1$ from $\tilde{q}(t)$ to $\tilde{x}(t)$ (as shown in (21)), and the complementary ISS gain $\gamma_2 = \beta$ from $\tilde{x}(t)$ to $\tilde{q}(t)$, we claim that when condition (10) is satisfied, that is when

$$\gamma_1 \cdot \gamma_2 = \beta \cdot \frac{1}{(1-\lambda)\alpha} < 1, \tag{31}$$

for some constant $0 < \lambda < 1$, the signals $\tilde{x}(t)$ and $\tilde{q}(t)$are $L_\infty$, and specifically

$$\|\tilde{x}(t)\|_{L_\infty} \le \mu\left(\|\tilde{x}(0)\| + \|\psi_1\|_{L_\infty} + \|\psi_2\|_{L_\infty}\right) \tag{32}$$

$$\|\tilde{x}\|_a = 0, \tag{33}$$

for some class - $\kappa$ function $\mu(\cdot)$. The claim follows from the same arguments as in the ISS Small-Gain Theorem [12], [13]. Because both $\psi_1$ and $\psi_2$ depend on $x(t)$ in the interval $[-\tau, 0]$, we can render $\|\psi_1\|_{L_\infty}$ and $\|\psi_2\|_{L_\infty}$ as small as desired by choosing $\psi(t)$, the initial condition of $x(t)$ in (7), sufficiently small. This stability argument, combined with the convergence property (33), proves global asymptotic stability for $x = x^*$.

## III. A New Class of Edge-Based AQM Algorithms

In this section, we apply the small-gain analysis in the previous section to a new edge-based AQM design. We consider a broader network scenario where the general utility function $U_i(x_i)$ and the pricing function $h_l(y_l)$ are employed in the source and the link controllers [1]:

$$\dot{x}_i = \kappa_i \left( U_i'(x_i) - q_i \right), \quad p_l = h_l(y_l) \tag{34}$$

The objective is to keep the steady-state queue-length at each router equal to zero besides the optimization in [1]. To achieve this goal, the aggregate rate $y_l$ at each link $l$ has to be regulated slightly less than the link capacity $c_l$, as in the adaptive virtual queue [8]. However, instead of paying to manage the queue at every router, we aim to realize AQM at the edge of the network, thus providing a cheaper and more robust alternative.

In this edge-based AQM, we assume that edge router is able to acquire the notional minimum path capacity $C_I$ and the corresponding path demand $D_I$ for path $I$, which is the series of bottlenecks between the ingress and egress router. Here we manage the flows in the same edge-to-edge path $I$ since all these flows contribute a virtual demand to this virtual link which will build up a virtual queue leading to a notional congestion signal. Here, the minimum path capacity $C_I$ is defined as $C_I = \min\limits_{l \in T(I)} \{c_l\}$, where $T(I)$ is the set of links which path $I$ traverses, while the path demand $D_I$ is artificially defined as the difference between path capacity and available bandwidth $\min\limits_{l \in T(I)} \{c_l - y_l\}$ on the path $I$:

$$D_I = \min_{l \in T(I)} \{c_l\} - \min_{l \in T(I)} \{c_l - y_l\} \tag{35}$$

For real networks, the information on $C_I$ and $D_I$ requires robust and short-time-scale estimation (e.g. [14]). However, we want to emphasize that we do not restrict our attention to any specific active measurements in this paper but propose a general design framework, which may leverage future advances in the detection area. In this scenario, if the target demand $D_I^*$ equals to the virtual path capacity, which we defined as $\gamma C_I$ with $0 < \gamma < 1$ being the desired *utilization* for the network, then the steady-state queue-length for path $I$ will be zero. This is because

$$D_I^* = \gamma C_I \Leftrightarrow \min_{l \in T(I)} \{c_l - y_l^*\} = (1 - \gamma) \min_{l \in T(I)} \{c_l\}, \tag{36}$$

that is, the real demand at any link in that particular path is less than its capacity $y_l \le c_l, \quad \forall l \in T(I)$. The merit is then that, the steady-state queue-length will be zero and there will be no queuing delay.

Under Kelly's optimization framework, (36) can be thought of as introducing the extra constraint

$$\min_{l \in T(I)} \{c_l - y_l^*\} \ge (1 - \gamma) \min_{l \in T(I)} \{c_l\} \tag{37}$$

to the static network utility optimization problem. Analogous to Kelly's primal design, we shall see that to fulfill this constraint, an extra price (implemented by dropping or marking) needs to be fed back to the source from edge router

$$\upsilon_I = h_e(D_I, \gamma C_I) \tag{38}$$

which, similar to $h_l(y_l)$ in (34), is a barrier function forcing $D_I$ to approach $\gamma C_I$. In the following design, we simply choose

$$h_e(D_I, \gamma C_I) = -\lambda (D_I - \gamma C_I) \tag{39}$$

with a sufficiently large $\lambda > 0$ and illustrate the global asymptotic stability of this design.

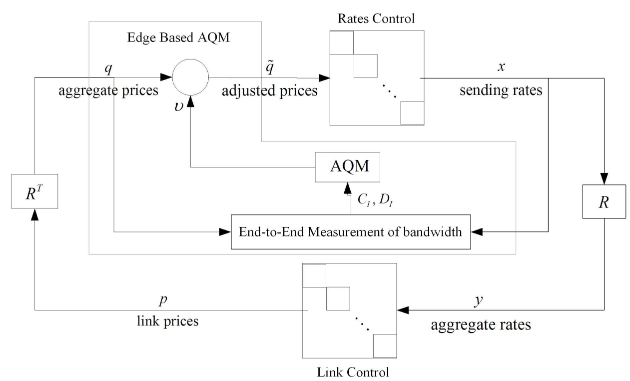


Fig. 2. Fluid network model For EVQ

For the purpose of illustration, in the following analysis, we assume that different users traverse in different paths and thus ignore the difference between notations $i$ and $I$, although the same derivation can be easily extended to the general case where multiple users share the same path $I$. We first denote $q$ as the overall price from the network and represent it as the product of an extended routing matrix $R_e$ and price $p_e$, where $R_e$ and $p_e$ are defined as follows:

$$q = R^T p + \upsilon = \underbrace{\begin{bmatrix} R^T & I_{N \times N} \end{bmatrix}}_{:= R_e^T} \underbrace{\begin{bmatrix} p \\ \upsilon \end{bmatrix}}_{:= p_e} = R_e^T p_e \tag{40}$$

$\upsilon$ models the extra regulation from edge router as defined in (38) and (39). To prove stability, we employ a similar but the more general logarithm transformation:

$$\begin{array}{l}\tilde{x}_i = \ln\frac{x_i+A}{x_i^*+A},\ \tilde{q}_i = \ln\frac{q_i+A}{q_i^*+A},\ \tilde{p}_l = \ln\frac{p_l+A}{p_l^*+A} \\ \tilde{y}_l = \ln\frac{y_l+A}{y_l^*+A},\ \tilde{\upsilon}_i = \ln\frac{\upsilon_i+A}{\upsilon_i^*+A}\end{array} \quad (41)$$

where $A > 0$ is sufficiently large such that $\upsilon_i + A$ and $\upsilon_i^* + A$ are positive for $\forall i$. In the proof of Theorem 1, $A$ is zero since the state variables $x_i$, $q_i$, $p_l$ and $y_l$ in (41) are all positive. Applying this general logarithmic state transformation (41), we next employ the Small Gain Theorem to show the global asymptotic stability of (34), (40). The ISS gains from $q_i$ to $x_i$ and from $y_l$ to $p_l$ of the source and the link controllers (34) have already been proved to be $\alpha$ and $\beta$ respectively in one of our earlier papers [15], where $\alpha > 0$ and $\beta > 0$ such that

$$U_i''(x_i) \le -\frac{1}{\alpha},\ \forall i, \quad \text{and}\ h_l'(y_l) < \beta,\ \forall l \quad (42)$$

We first claim that the ISS gains of the transformed system from $\tilde{q}_i$ to $\tilde{x}_i$ and from $\tilde{y}_l$ to $\tilde{p}_l$ equals $\alpha$ and $\beta$ respectively by employing the following proposition.

*Proposition 1:* Suppose two bounded signals $x(t)$ and $u(t)$ satisfy

$$\|x - x^*\|_a \le \alpha \|u - u^*\|_a \quad (43)$$

then

$$\left\|\ln\frac{x+A}{x^*+A}\right\|_a \le \alpha \left\|\ln\frac{u+A}{u^*+A}\right\|_a \quad (44)$$

for sufficiently large $A > 0$.

Next, we argue that the extra price $\upsilon$ in (39) from edge router, driven by $(D_I, \gamma C_I)$, implies a unit ISS gain from $\tilde{x}$ to $\tilde{\upsilon}$. One can show this by following similar arguments as in the proof of Theorem 1. It is important to note that this gain from $\tilde{x}$ to $\tilde{\upsilon}$ is independent of $\gamma$ because $\gamma$ is cancelled as in the logarithm transformation (11). Combining the ISS gain property of original price feedback $p$ from network, i.e., the gain $\beta$ from $\tilde{x}$ to $\tilde{p}$, we claim that the ISS gain from $\tilde{x}$ to $\tilde{p}_e$ is $\tilde{\gamma}_2 = \max\{1, \beta\}$. It further implies that the gain from $\tilde{x}$ to $\tilde{q}$ is $\tilde{\gamma}_2$ because of the unit gain from $\tilde{p}_e$ to $\tilde{q}$ from in (40). We then use the ISS small gain theorem to conclude the following result:

*Theorem 2:* Consider the source and the link controllers (34) with the edge-based AQM (40). The network is globally asymptotically stable if (42) is satisfied and

$$\alpha\tilde{\gamma}_2 = \alpha \max\{1, \beta\} < 1 \quad (45)$$

Note that Theorem 2 does not consider the transmission and queuing delays in the Theorem 1. However, as in the proof of Theorem 1, the conclusion still holds in the time-delayed model due to the fact that the delays do not introduce any extra gain to the system (time-delays have an amplitude of 1). That is, the ISS gain from $x_\tau$ to $x$ is equal to 1, so is the gain from $q_\tau$ to $q$. Thus, the global asymptotic stability holds for an arbitrary number of flows and any amount of time delays. We also note that through regulating $D_I$ at the edge router, we decouple the task of managing bottleneck queues and the placement of this management function. In other words, with this new design, we do not require any AQM components to be present at every bottleneck and essentially move this important function to network edges.

## IV. Application to CDMA Power Control

Recently, CDMA power control has been studied as a noncooperative game-theoretic optimization problem in [9], [10], where each user tries to maximize

$$\max_i\ J_i = U_i(\gamma_i(p)) - P_i(p_i), \quad (46)$$

in which

$$\gamma_i(p) =: \frac{Lh_ip_i}{\sum\limits_{k\neq i} h_kp_k + \sigma^2}, \quad i = 1, \cdots, M \quad (47)$$

is the signal-to-interference ratio (SIR), $L$ is the spreading gain, $h_i$ is the channel gain between the $i^{th}$ mobile and the base station, and $\sigma^2$ is the noise variance containing the contribution of the secondary background interference. In this formulation,

$$U_i(\gamma_i) = u_i \log(\gamma_i + L), \quad (48)$$

is the utility function for the $i^{th}$ user, which corresponds to the demand for bandwidth, and $P_i(p_i)$ corresponds to the cost of power. As shown in [9], [10], when $P_i(p_i)$ is twice continuously differentiable, non-decreasing, and strictly convex, (46) has a unique Nash equilibrium, and the network converges to this equilibrium if the mobiles implement the gradient update law

$$\begin{aligned}\dot{p}_i &= -\lambda_i\frac{\partial J_i}{\partial p_i} = \frac{dU_i}{d\gamma_i}\frac{L\lambda_ih_i}{\sum_{k\neq i} h_kp_k+\sigma^2} - \lambda_i\frac{dP_i(p_i)}{dp_i} \\ &= -\lambda_i\frac{dP_i(p_i)}{dp_i} + \frac{u_i\lambda_ih_i}{\sum_k h_kp_k+\sigma^2}, \quad \lambda_i > 0.\end{aligned} \quad (49)$$

To show that the system (49) indeed conforms to the structure (34), we let $M$ be the number of the mobiles,

$$R =: \begin{bmatrix} h_1 & h_2 & \cdots & h_M \end{bmatrix}^T \quad (50)$$

$$q =: \varphi(y) = -\frac{1}{y + \sigma^2} \quad (51)$$

$$y =: h^T p \quad (52)$$

$$w =: -h \cdot q \quad (53)$$

and represent (49) as in (34), where the feedforward block is

$$\dot{p}_i = -\lambda_i\frac{dP_i(p_i)}{dp_i} + u_i\lambda_iw_i, \quad i = 1, \cdots, M. \quad (54)$$

In this representation, the forward block corresponds to the mobiles and the feedback block corresponds to the base station. Using the same equilibrium shift as in (11) and with the help of Theorem 1, we obtain the following result:

*Theorem 3:* Consider the feedback interconnection (50)-(54), and suppose that $P_i(p_i)$ is

$$P_i(p_i) = \frac{p_i^{1-m}}{1-m}, \quad m > 0, \quad \forall i = 1, \cdots, N \quad (55)$$

Then, the equilibrium $p = p^*$ is globally asymptotically stable for arbitrary delays if $m > 1$.

This theorem follows because, $\varphi(\cdot)$ implies

$$\left|\ln \frac{q}{q^*}\right| = \left|\ln \frac{1}{y+\sigma^2} - \ln \frac{1}{y^*+\sigma^2}\right| \leq \left|\ln \frac{y}{y^*}\right|, \quad (56)$$

which means the ISS gain from $\tilde{y}$ to $\tilde{q}$ for the price generation subsystem is equal to 1. Thus, the small gain condition is satisfied when $m > 1$.

## V. CONCLUSIONS

We have studied the stability problem of a general network with both transmission and queuing delays and arbitrary routing matrices. In particular, we have presented a novel small gain approach to show the global asymptotic stability of such a network. More specifically, we have utilized the logarithmic state transformation recently proposed in literature and proved that the stability condition is scalable and independent of the routing matrices based on the new state variables. Unlike the previously reported results that employ the logarithmic transformation, we have given a simple small-gain interpretation for the delay robustness of the network. We have also illustrated the general applicability of the analysis by extending the study to the AQM problem in the Internet and the power control problem in CDMA systems.

## REFERENCES


[1] F. Kelly, A. Maulloo, and D. Tan. Rate control in communication networks: shadow prices, proportional fairness and stability. Journal of the Operational Research Society, 49:237–252, 1998.

[2] S.H. Low and D.E. Lapsley. Optimization flow control – I: basic algorithm and convergence. IEEE/ACM Transaction on Networking, 7(6):861–874, 1999.

[3] S. Kunniyur and R. Srikant. End-to-end congestion control: Utility functions, random losses and ecn marks. In Proceedings of INFOCOM 2000, Tel Aviv, Israel, March 2000.

[4] G. Vinnicombe. On the stability of networks operating TCP-like congestion control. In Proceedings of the IFAC World Congress, Barcelona, Spain, 2002. University of Cambridge Technical Report CUED/F-INFENG/TR.398. Available at http://www.eng.cam.ac.uk/gv.

[5] X. Fan, Robust nonlinear control in communication networks, PhD Dissertation, ECSE, RPI , Dec. 2004.

[6] L. Ying, G. Dullerud and R. Srikant. Global Stability of Internet Congestion Controllers with Heterogeneous Delays. To appear in the IEEE/ACM Transactions on Networking. An earlier version appeared in the Proc. American Control Conference, 2004.

[7] E. Sontag. Smooth stabilization implies coprime factorization. IEEE Transactions on Automatic Control, 34:435-443, 1989.

[8] Kunniyur and R. Srikant. Analysis and design of an adaptive virtual queue algorithm for active queue management. In Proceedings ACM Sigcomm, 123–134, San Diego, CA, Aug. 2001.

[9] T. Alpcan, T. Basar, R. Srikant, and E. Altman. CDMA uplink power control as a noncooperative game. Wireless Networks, 8:659–669, November 2002.

[10] T. Alpcan and T. Basar. A Hybrid Systems Model for Power Control in Multicell Wireless Networks. In Proceedings of WiOpt'03: Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks, INRIA Sophia-Antipolis, France, March 3-5, 2003.

[11] X. Fan, M. Arcak, and J.T. Wen. Passivation Designs for CDMA Uplink Power Control. In Proceedings of the American Control Conference, Boston, MA, June 2004.

[12] Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. IEEE Transactions on Automatic Control, 41(9):1256–1271, 1996.

[13] Z.-P. Jiang, and A. Teel, and L. Praly. Small-Gain Theorem for ISS Systems and Applications. Mathematics of Control, Signals, and Systems, 7:95-120, 1994.

[14] V. Ribeiro, R. Riedi, R. Baraniuk, J. Navratil, and L. Cottrell. PathChirp: Efficient Available Bandwidth Estimation for Network Paths. Passive and Active Measurement Workshop, 2003.

[15] X. Fan, M. Arcak, J. T.Wen. Robustness of Network Flow Control Against Disturbances and Time-Delay. Systems & Control Letters, 53(1):13-29.